The Theory and Application of the 'Curve Constant in Curve Ranging'

Oliver C. Ojinnaka*

Abstract

The application of modern surveying techniques has greatly reduced the rigours associated with engineering surveys. The known properties of curves have also been incorporated in these techniques in the setting-out operations, and accuracy is usually enhanced by applying arc-to-chord correction. It has however been difficult to differentiate between the errors due to field operation and those due to the curve-ranging technique employed. A new and unique property of circular curves known as Curve Constant (CC) has been developed in this work. The CC is employed to facilitate planning in curve ranging and also serves as an aid in quality control for setting out. The theory is developed from the principles of the Optimum Point Method of curve ranging, and the derivation of the CC by both empirical and theoretical methods are discussed. A simplified version of the formula is given at the end for ease of computation. The highlighted applications of the CC confirm it as an invaluable tool for planning in curve-ranging operations.

1.0 Introduction

Curve ranging has remained an integral part of the route surveyors' assignments for decades. Several techniques of curve ranging have been exhaustively discussed by previous authors (Meyer, 1969 and Raymond, 1981) In addition to the several existing techniques of curve ranging, a new method known as the Optimum Point Method (OPM) has been developed (Ojinnaka and Ndukwe, 1989). The application of the OPM in curve ranging for spirals was also discussed by Ndukwe and Ojinnaka (1991). In the computations for the curve-points coordinates, misclosure is observed between the correct and computed values of the last tangent point.

In most of the earlier methods of curve ranging, it was difficult to distinguish errors of observation from those inherent in the theory of the applied technique. This newly developed property of curve is employed for the following:

(i) To separate the actual error in setting out due to the technique adopted from the error due to field operations;

* Department of Geoinformatics and Surveying, University of Nigeria

(ii) To determine the relationship between the calculated error of misclosure and the arc length adopted in the setting out; and

(iii) To examine the application of the above relationship as an aid to office planning for curve ranging operations.

A summary of the OPM theory is given below. Interested readers are therefore advised to consult the given references for further reading..

2.0 Curve Ranging by the Optimum Point Method (OPM)

Figure 1 illustrates the principles of OPM. The basic setting-out technique employs radial angle (θ) and chord length (1) or radial distances (s). The radial angles and distances are calculated from the known coordinates of the curve-points (CP). The coordinates of the curve-points are computed by running a traverse on the curve system with a local origin at the midpoint of the long chord. This origin is designated as optimum point (Ojinnaka and Ndukwe, 1989). The traverse is run from the Optimum Point (Z) through A, b, c, .. B, Z, with ZA as the direction of the y-axis or local north.

3.0 Theory of the Curve Constant

A sample of calculations for curve ranging by OPM is given in Appendix 1. It can be observed in columns 4 and 5 that the errors in x-axis (dx) and y-axis (dy) are 0.000 and 0.010m respectively. This gives a misclosure (e) of 0.010m between last curve-point and the point-of-tangency (PT) where e is given by $e = (dx^2 + dy^2)^{1/2}$. Appendix 2 shows computations for the same curve repeated with chord lengths of 40m. It is observed that the misclosure increased to a value of 0.040m. Subsequent variations of chord length enable us to obtain some relationships between chord lengths and the corresponding misclosures. Appendix 3 gives values of errors (e) for given values of chord lengths (1). From Appendix 3, it can be observed that a misclosure of 0.01m was obtained with 20m chord lengths whereas a value of 0.040 (4 times 0.010) was obtained for the same curve when 40m (or 2 times 20m) was adopted. Denoting the chords of 20m and 40m by 1_1 and 1_2 respectively and their corresponding errors by e_1 and e_2, we can write:.

$$\frac{l_1}{l_2} = \frac{1}{2} \quad : \quad \frac{e_1}{e_2} = \frac{0.010}{0.040} = \frac{1}{4}$$

$$\therefore \quad \frac{e_1}{e_2} = \left(\frac{l_1}{l_2}\right)^2 \qquad (1)$$

Further investigation with the other values in Table 3 shows that for any pair of chord lengths considered, there exists a unique relationship (given by equation 1) between the chord lengths 1_1 and 1_2 and the respective errors e_1 and e_2. This holds true for any pair of chord lengths adopted, and minor discrepancies may be merely due to rounding off errors. We can rewrite equation (1) in the form:

$$l_2 = l_1\left(\frac{e_2}{e_1}\right)^{\frac{1}{2}} \tag{2}$$

3.1 Determination of the Curve Constant

Assuming initial values of 1_1 and e_1 have been obtained from calculation, and subsequent values of e_i or 1_i are required, then from equation (2) we have:

$$l_i = l_1\left(\frac{e_i}{e_1}\right)^{\frac{1}{2}} = \frac{\sqrt{e_i}}{\sqrt{e_1}/l_1} \tag{3}$$

But since e_1 and 1_1 are constants in subsequent calculation, we write equation (3) as:

$$l_i = \frac{\sqrt{e_i}}{k} \tag{4}$$

where $k = \frac{\sqrt{e_i}}{l_i}$ (5)

That is, $l_i \propto \sqrt{e_i}$ *or* $e_i = (l_i k)^2$ (6)

Equation (6) is applied to determine the error when a given chord length is used while equation (4) gives the chord length required to achieve a specified accuracy in curve ranging. From equation (4), we also have that:

$$k = \frac{\sqrt{e_1}}{l_1} \tag{7}$$

Equation (7) implies that for any given curve, the ratio of the square root of the error and the chord length is a constant irrespective of the length of the chord length adopted. The third row of appendix 3 shows the values of k obtained from equation (7) using the values of 1_i and e_i. This, therefore, confirms the validity of equation (7). The value k is subsequently referred to as the "Curve Constant".

3.2 Derivation of the "Curve Constant"

The values of the curve constants can be derived either empirically or theoretically as discussed below:

3.3 Empirical Determination of the Curve Constant

Appendix 3 gives an example of empirical determination. In this method, it is recommended that for simplicity and speed, the curve should be divided into two equal halves. The length of the ensuing arc is then used for the computations, as already described in appendix 1, without any need to include the columns for radial distances and radial angles since they are not needed.

3.4 Theoretical Derivation of K

Theoretically, the curve constant can be derived from the first principles. In approaching this method, looking at appendix 1, it is observed that while column 4 shows an error of 0.010m for y-axis, column 5 shows no error for the x-axis. Due to the symmetry of the circular curve, the errors in the axis cancel out between the tangent points thereby leaving the errors in the y-axis. Consequently, the error in the computation of the last curve point accumulates only on the y-axis. To facilitate the computation, the circular curve, in figure 2, is divided into even number of equal chords.

With reference to appendix 2 and from the preceding arguments, the value of the error (e) from the equations below can be obtained.

i e. $$Aa' + bb' + cc' + dd' + ee' + ff' - AB = e \quad (8)$$

or $$m\text{-}AB = e \quad (9)$$

where m = sum of absolute values of the differences on y-axis.

If the length of the curve 'L' is divided into N number of equal arcs, then the length of each arc will be given by

$$l = L/N$$

Therefore, from equation (7),

$$k = \frac{\sqrt{e}}{l} = \frac{N\sqrt{e}}{L} \quad (10a)$$

but L = RI

where

R = radius of curvature
I = Intersection angle in radians (angle between the tangents)

$$k = \frac{N\sqrt{e}}{RI} \quad m^{-1/2} \quad (10b)$$

However, since the curve is divided into even number of equal arcs, and since circular curve is symmetrical, it follows that:

$$Aa' + bb' + cc' = dd' + ee' + ff'$$

Therefore, the value of 'm' contained in equation (9) becomes

$$M = 2\,(Aa' + bb' + cc') \quad (11)$$

let Aa'= r., bb'= r_1, cc' = r_2 ..... etc. as shown in figure 2

Then m= $2(r_o + r_1 + r_2 + ... + r_n)$, where n = N/2-1

For circular curves, the angles at the curve-points b, c, d ... are equal and given by

$$a = \frac{I}{N} \quad (12)$$

The angle at A (the first deflection angle) is given by

$$\alpha = \frac{a}{2} = \frac{I}{2N} \quad (13)$$

But the differences in y-axis $r_o, r_1, r_2, \ldots r_n$ are given by

$$\begin{aligned} r_o &= l\cos\varphi_o \\ r_1 &= l\cos\varphi_1 \\ R_2 &= l\cos\varphi_2 \\ &\text{--------------} \\ &\text{--------------} \\ r_n &= l\cos\varphi_n \end{aligned} \quad (14)$$

where

$\varphi, \varphi_1, \varphi$ ---- φ_n = angles between the chords and the long chord AB (see figure 2)

but
$$\begin{aligned} \varphi_o &= I/2 - \alpha \\ \varphi_1 &= \varphi_o - 2\alpha \\ \varphi_2 &= \varphi_o - 4\alpha \\ &\text{------------} \\ &\text{--------------} \\ \varphi_n &= \varphi_o \; 2\alpha \end{aligned}$$

Therefore, from equations (11) and (14),

$$m = 2l(\cos\varphi + \cos(\varphi_o - 2\alpha) + \cos(\varphi_o - 4\alpha) + \ldots.. + \cos(\varphi_o - 2_n\alpha))$$

but $l = L/N = RI_r / N$ and $\alpha = 1/2N$

where I_r is the value of I in radians.

$$\therefore \; m = \frac{2RI_r}{N}\left(\cos\frac{I(N-1)}{2N} + \cos\frac{I(N-3)}{2N} + \ldots\ldots + \cos\frac{I(N-1-2n)}{2N}\right)$$

$$i.e. \; m = \frac{2RI_r}{N}\sum_{i=0}^{N/2-1}\cos\frac{I(N-1-2i)}{2N} \quad (15)$$

The long chord AB is given by

$$AB = 2R\sin\frac{I}{2} \tag{16}$$

Therefore, from equations (9), (15) and (16), the value for the error 'e' becomes

$$e = m - AB = \frac{2RI_r}{N}\sum_{i=0}^{N/2-1}\cos\frac{I(N-1-2i)}{2N} - 2R\sin\frac{I}{2} \tag{17}$$

Hence, from equation (10),

$$K = \frac{Ne^{1/2}}{RI_r} = \frac{N\left\{\frac{2I_r}{N}\sum_{i=0}^{N/2-1}\cos\frac{I(N-1-2i)}{2N} - 2\sin\frac{I}{2}\right\}^{1/2}}{R^{1/2}I_r} \tag{18}$$

In order to verify the validity of equation (18), values of K were computed for different values of N using the same data which was employed in the computation of the curve in appendix 3. The following values of K were obtained for the corresponding values of N.

N:	2	4	6	8	10	
K:	5	5	5	5	5	$\times 10^{-3} m^{-1/2}$

The above values of N correspond to chord lengths (l) of 100m, 50, 33.3, 25 and 20m respectively. It is observed that the values of k are the same as those computed empirically in Appendix 3.

Equation (18) can be written in a simplified form as:

$$K = CR^{\frac{1}{2}} \tag{19}$$

where

$$C = \frac{N\left\{\frac{2I_r}{N}\sum_{i=0}^{N/2-1}\cos\frac{I(N-1-2i)}{2N} - 2\sin\frac{I}{2}\right\}^{1/2}}{I_r} \tag{20a}$$

But R is a constant for a given circular curve, and since K is a constant, it then follows that 'C' is also a constant. Hence, k and R can be related by a proportionality equation given by:

$$K \propto R^{-1/2} \tag{20b}$$

C is the constant of proportionality and is subsequently referred to as 'curve factor'. From the above derivations, it can be inferred that the curve constant varies